

# Ingredients of supergravity

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These notes give a summary of lectures given in Corfu in 2010 on basic ingredients in the study of supergravity. It also summarizes initial chapters of a forthcoming book ‘Supergravity’ by the same authors.

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The lectures on ‘Supergravity’ in Corfu were inspired by a forthcoming book [1]. They gave a summary of the first chapters of that book.

The main ingredients of supersymmetry can be seen from the first supersymmetric field theory as written in [2, 3]. A scalar field  $A(x)$  transforms into a fermion  $\psi(x)$  with a spinor parameter  $\epsilon$ :

$$\delta(\epsilon)A(x) = \bar{\epsilon}\psi(x). \quad (0.1)$$

Since scalars have engineering dimension 1, and fermions have dimension 3/2, the parameter should have dimension  $-1/2$ . Therefore in the transformation of the fermion into the boson, one should have a derivative (if no negative dimension objects are introduced). The transformation should thus be of the form

$$\delta(\epsilon)\psi(x) = \gamma^\mu \epsilon \partial_\mu A(x). \quad (0.2)$$

Details will be discussed later, but the general form leads to the idea that the commutator of two supersymmetry transformations is a translation

$$[\delta(\epsilon_1), \delta(\epsilon_2)] = \bar{\epsilon}_2 \gamma^\mu \epsilon_1 \partial_\mu, \quad (0.3)$$

or with  $Q$  the operator of supersymmetry, and  $P_\mu$  the one of translations, this gives a relation of the form  $\{Q, Q\} = \gamma^\mu P_\mu$ .

The general philosophy of the successes of field theory in the 70’s was the idea that symmetries should be promoted to local symmetries. When this is done with the supersymmetry algebra, the gauge theory of translations appears and therefore the theory contains gravity, i.e. *supergravity* [4]. Supergravity is a basic tool in the study of string theory. The AdS/CFT ideas, allow to study non-perturbative field theories based on dualities with supergravity solutions. Phenomenological models in high-energy physics are developed as supergravity theories based on compactifications on Calabi-Yau manifolds. Also many cosmological models use a supergravity limit of string theory. Supergravity allows also to study objects like black holes, cosmic strings, domain walls, ..., as solutions of field equations of local supersymmetric actions.

## 1 Scalar field theory and its symmetries

Transformations of the Poincaré group act on spacetime points as

$$x^\mu = \Lambda^\mu{}_\nu x'^\nu + a^\mu. \quad (1.1)$$

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We are mainly concerned with infinitesimal transformations and thus expand the parameters of the Lorentz transformations as

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \lambda^\mu{}_\nu + \mathcal{O}(\lambda^2). \quad (1.2)$$

The generators related to the parameters  $\lambda^{\mu\nu}$  satisfy the algebra (we use the metric with signature  $(- + \dots +)$ )

$$[m_{[\mu\nu]}, m_{[\rho\sigma]}] = \eta_{\nu\rho} m_{[\mu\sigma]} - \eta_{\mu\rho} m_{[\nu\sigma]} - \eta_{\nu\sigma} m_{[\mu\rho]} + \eta_{\mu\sigma} m_{[\nu\rho]}. \quad (1.3)$$

This algebra is satisfied by the differential operators

$$L_{[\rho\sigma]} \equiv x_\rho \partial_\sigma - x_\sigma \partial_\rho. \quad (1.4)$$

These are used in the definitions of the transformations of scalars using rule  $\phi(x) = \phi'(x')$ :

$$\phi(x) \rightarrow \phi'(x) = U(\Lambda)\phi(x) = \phi(\Lambda x + a), \quad U(\Lambda) \equiv e^{-\frac{1}{2}\lambda^{\rho\sigma} L_{[\rho\sigma]}}. \quad (1.5)$$

For fields that are not scalars, Lorentz transformations require matrices  $m_{[\mu\nu]}$  that act on the different components and the full transformation is

$$\psi(x) \rightarrow \psi'(x) = U(\Lambda)\psi(x) = e^{-\frac{1}{2}\lambda^{\rho\sigma} m_{[\rho\sigma]}} \psi(\Lambda x + a). \quad (1.6)$$

In general, symmetries with constant parameters  $\epsilon^A$  can be written using operators  $\Delta_A$ , such that

$$\delta\phi^i(x) \equiv \epsilon^A \Delta_A \phi^i(x). \quad (1.7)$$

The Lagrangian is not necessarily invariant, but can transform into a total derivative

$$\delta\mathcal{L} \equiv \epsilon^A \left[ \frac{\delta\mathcal{L}}{\delta\phi^i} \partial_\mu \Delta_A \phi^i + \frac{\delta\mathcal{L}}{\delta\phi^i} \Delta_A \phi^i \right] = \epsilon^A \partial_\mu K_A^\mu. \quad (1.8)$$

This leads to conserved currents (using the Euler-Lagrange equations as indicated by  $\approx 0$ )

$$J^\mu{}_A = -\frac{\delta\mathcal{L}}{\delta\partial_\mu \phi^i} \Delta_A \phi^i + K_A^\mu, \quad \partial_\mu J^\mu{}_A \approx 0. \quad (1.9)$$

## 2 The Dirac field

The Dirac field equation is

$$\not{\partial}\Psi(x) \equiv \gamma^\mu \partial_\mu \Psi(x) = m\Psi(x), \quad (2.1)$$

where the defining equation for gamma matrices is

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu} \mathbf{1}. \quad (2.2)$$

For Lorentz transformations the matrix  $m_{[\mu\nu]}$  in (1.6), which satisfies the Lorentz algebra, is

$$\Sigma^{\mu\nu} \equiv \frac{1}{4} [\gamma^\mu, \gamma^\nu]. \quad (2.3)$$

### 3 Clifford algebras and spinors

We study in this chapter the gamma matrices that satisfy (2.2) in various spacetime dimensions. These determine the properties of the spinors in the theory and of the supersymmetry algebra. We first want to know the size of the smallest spinors in each dimension, whether they can be chosen to be real. Furthermore we want to know which spinor bilinears are symmetric or antisymmetric in the two spinors. The latter is important since they will occur in any superalgebra, similar to (0.3). For the latter to be consistent we need

$$\bar{\epsilon}_1 \gamma^\mu \epsilon_2 = -\bar{\epsilon}_2 \gamma^\mu \epsilon_1. \quad (3.1)$$

We always use gamma matrices that satisfy

$$\gamma^{\mu\dagger} = \gamma^0 \gamma^\mu \gamma^0. \quad (3.2)$$

Thus for spacelike  $\mu$  they are Hermitian. For even dimensions  $D = 2m$ , we define

$$\gamma_* \equiv (-i)^{m+1} \gamma_0 \gamma_1 \dots \gamma_{D-1}, \quad (3.3)$$

which satisfies  $\gamma_*^2 = \mathbf{1}$ . E.g. for  $D = 4$  we have  $\gamma_* = i\gamma_0 \gamma_1 \gamma_2 \gamma_3$ . These are used to split spinors in left-handed and right-handed ones using projection operators

$$P_L = \frac{1}{2}(\mathbf{1} + \gamma_*), \quad P_R = \frac{1}{2}(\mathbf{1} - \gamma_*). \quad (3.4)$$

The full Clifford algebra also contains gamma matrices that are antisymmetric in multiple indices. They are defined by

$$\gamma^{\mu_1 \dots \mu_r} = \gamma^{[\mu_1} \dots \gamma^{\mu_r]}, \quad \text{e.g.} \quad \gamma^{\mu\nu} = \frac{1}{2} \gamma^\mu \gamma^\nu - \frac{1}{2} \gamma^\nu \gamma^\mu, \quad (3.5)$$

Since symmetries of spinor bilinears as in (3.1) are important for supersymmetry, we use the Majorana conjugate to define  $\bar{\lambda}$

$$\bar{\lambda} \equiv \lambda^T C. \quad (3.6)$$

$C$  is a matrix such that  $C\gamma_{\mu_1 \dots \mu_r}$  are all symmetric or antisymmetric, depending only on  $D$  (modulo 8) and  $r$  (modulo 4):

$$\bar{\lambda} \gamma_{\mu_1 \dots \mu_r} \chi = t_r \bar{\chi} \gamma_{\mu_1 \dots \mu_r} \lambda, \quad t_r = \pm 1. \quad (3.7)$$

For general spinors (3.6) differs from the Dirac adjoint  $\bar{\Psi} \equiv \Psi^\dagger i\gamma^0$ , but we will see below that it agrees for certain types of spinors that are called Majorana spinors. The values of  $t_r$  are constrained by consistency conditions in any dimension. E.g. for  $D = 2, 3, 4 \bmod 8$ , which we will mainly use in these lectures one can use  $t_0 = t_3 = 1, t_1 = t_2 = -1$ . These sign factors also determine the adjoint of composite expressions of spinors and gamma matrices, i.e.

$$\chi = \Gamma^{(r_1)} \Gamma^{(r_2)} \dots \Gamma^{(r_p)} \lambda \implies \bar{\chi} = t_0^p t_{r_1} t_{r_2} \dots t_{r_p} \bar{\lambda} \Gamma^{(r_p)} \dots \Gamma^{(r_2)} \Gamma^{(r_1)}. \quad (3.8)$$

When we need spinor indices, we contract them always in NW-SE convention, using  $\mathcal{C}^{\alpha\beta}$ , which are the components of  $C^T$  and  $\mathcal{C}_{\alpha\beta}$ , which are the components of  $C^{-1}$ , such that  $\lambda_\alpha$  are the components of a spinor  $\lambda$  and  $\lambda^\alpha$  are those of  $\bar{\lambda}$ :

$$\lambda^\alpha = \mathcal{C}^{\alpha\beta} \lambda_\beta, \quad \lambda_\alpha = \lambda^\beta \mathcal{C}_{\beta\alpha}. \quad (3.9)$$

The components of ordinary gamma matrices are written as  $(\gamma_\mu)_\alpha{}^\beta$ , whose indices can be raised or lowered with the same rules to get to symmetric or antisymmetric matrices:

$$(\gamma_\mu)_{\alpha\beta} = (\gamma_\mu)_\alpha{}^\gamma \mathcal{C}_{\gamma\beta} = -t_1 (\gamma_\mu)_{\beta\alpha}. \quad (3.10)$$

Complex conjugation can be replaced by an operation called charge conjugation. The latter acts as complex conjugation on scalars, and has a simple action on fermion bilinears.<sup>1</sup> For example, it preserves the order of spinor factors. For all practical purposes one can consider the charge conjugate  $\lambda^C$  as the complex conjugate of the spinor  $\lambda$ . For a spinor bilinear, using a matrix in spinor space  $M$ , we have

$$(\bar{\chi} M \lambda)^* \equiv (\bar{\chi} M \lambda)^C = (-t_0 t_1) \bar{\chi}^C M^C \lambda^C. \quad (3.11)$$

Thus one only has to know the charge conjugate of matrices in spinor space, e.g.

$$(\gamma_\mu)^C = (-t_0 t_1) \gamma_\mu, \quad (\gamma_*)^C = (-)^{D/2+1} \gamma_*. \quad (3.12)$$

A priori a spinor has  $2^{\text{Int}[D/2]}$  (complex) components. We saw already that for even dimensions they can be reduced by a factor 2 using the projections (3.4). These define *Weyl spinors*. In some dimensions (and spacetime signature, but we will always assume Minkowski signature here) there are reality conditions  $\psi = \psi^C$ , consistent with Lorentz algebra. This defines *Majorana spinors*. The consistency condition for this definition can be expressed in terms of the sign factors in (3.7) as  $t_1 = -1$ . With (3.4) and (3.12) it is easy to see that such a reality condition can only be consistent with a chiral projection if  $D = 2 \bmod 4$  (and due to  $t_1 = -1$  in fact  $D = 2 \bmod 8$ ). When we can define real chiral spinors, we call them *Majorana-Weyl*. This leads in  $D = 10$  to spinors with only 16 real components. In  $D = 4$  both Majorana spinors and Weyl spinors have 4 real components. It is equivalent to write fermions in terms of Majorana spinors  $\psi$  or in terms of the Weyl spinors  $P_L \psi$ :

$$(P_L \psi)^C = P_R \psi, \quad (P_R \psi)^C = P_L \psi. \quad (3.13)$$

In dimensions with  $t_1 = 1$  one can define reality conditions for doublets of spinors

$$\chi^i = \varepsilon^{ij} (\chi^j)^C. \quad (3.14)$$

This defines *symplectic Majorana-Weyl* spinors. But this does in fact not diminish the minimal number of real components. E.g. for  $D = 5$  the minimal spinor has either as Dirac spinor or as symplectic-Weyl spinor 8 real components.

## 4 The Maxwell and Yang-Mills Gauge Fields

Maxwell fields with gauge transformations  $\delta A_\mu = \partial_\mu \theta(x)$  couple typically to complex fields like spinors that transform like  $\delta \psi = i q \theta \psi$ . The standard actions are then of the form

$$\begin{aligned} S[A_\mu, \bar{\Psi}, \Psi] &= \int d^D x \left[ -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \bar{\Psi} (\gamma^\mu D_\mu - m) \Psi \right], \\ D_\mu \Psi &\equiv (\partial_\mu - i q A_\mu) \Psi, \quad F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu. \end{aligned} \quad (4.1)$$

In many supersymmetric theories several Abelian gauge vectors  $A_\mu^A$ ,  $A = 1, \dots, m$ , appear and the generalized electromagnetic duality transformations play an important role. These apply in general actions for  $D = 4$  of the form

$$\mathcal{L} = -\frac{1}{4} (\text{Re } f_{AB}) F_{\mu\nu}^A F^{\mu\nu B} + \frac{1}{8} \varepsilon^{\mu\nu\rho\sigma} (\text{Im } f_{AB}) F_{\mu\nu}^A F_{\rho\sigma}^B, \quad \varepsilon_{012(D-1)} = 1, \quad \varepsilon^{012(D-1)} = -1, \quad (4.2)$$

where  $f_{AB}$  is a complex matrix that may depend on scalar fields in the theory. Using notations with (anti)self-dual tensors

$$F_{\mu\nu}^{\pm A} \equiv \frac{1}{2} (F_{\mu\nu}^A \pm \tilde{F}_{\mu\nu}^A), \quad \tilde{F}^{\mu\nu} = -\frac{1}{2} i \varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}^A, \quad (4.3)$$

<sup>1</sup> In terms of complex conjugate, the charge conjugate of a spinor is  $\lambda^C \equiv i t_0 \gamma^0 C^{-1} \lambda^*$ .

the Bianchi identities and field equations can be written in a similar form:

$$\partial^\mu \text{Im } F_{\mu\nu}^{A-} = 0, \quad \partial_\mu \text{Im } G_A^{\mu\nu-} = 0, \quad (4.4)$$

where

$$G_A^{\mu\nu} \equiv \varepsilon^{\mu\nu\rho\sigma} \frac{\delta S}{\delta F_{\rho\sigma}^A}, \quad G_A^{\mu\nu-} = i f_{AB} F^{\mu\nu-B}. \quad (4.5)$$

A priori, the equations (4.4) seem to be invariant under general real linear transformations

$$\begin{pmatrix} F'^{-} \\ G'^{-} \end{pmatrix} = \mathcal{S} \begin{pmatrix} F^{-} \\ G^{-} \end{pmatrix} \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} F^{-} \\ G^{-} \end{pmatrix}. \quad (4.6)$$

The compatibility with (4.5) requires that  $\mathcal{S}$  is a symplectic matrix, and that simultaneously the matrix  $f_{AB}$  transforms as

$$i f' = (C + i D f)(A + i B f)^{-1}. \quad (4.7)$$

We thus conclude that the duality transformations in 4 dimensions are transformations in the symplectic group  $\text{Sp}(2m, \mathbb{R})$ . [5]

## 5 The free Rarita-Schwinger field

Massless spin 3/2 fields are described by spinor-vectors  $\Psi_\mu(x)$  that transform with a local spinorial parameter as  $\delta\Psi_\mu(x) = \partial_\mu \epsilon(x)$ . The action that gives the right propagating degrees of freedom is

$$S = - \int d^D x \bar{\Psi}_\mu \gamma^{\mu\nu\rho} \partial_\nu \Psi_\rho. \quad (5.1)$$

Its field equation,  $\gamma^{\mu\nu\rho} \partial_\nu \Psi_\rho = 0$  is equivalent to

$$\gamma^\mu (\partial_\mu \Psi_\nu - \partial_\nu \Psi_\mu) = 0, \quad (5.2)$$

and after choosing a gauge fixing leads to solutions in terms of  $(D-3)2^{[\frac{D}{2}]}$  initial conditions.

It is useful to remind that there are two countings of the number of degrees of freedom (dof) of fields:

**on-shell counting:** is the number of helicity states, and can be obtained by counting the number of independent initial conditions for the field equations divided by 2 (since these are coordinates and momenta).

**off-shell counting:** is the number of field components – the number of gauge transformations.

While real scalars have 1 dof in both countings, a real spinor has  $2^{[D/2]}$  off-shell, and half of these on-shell dof. A massless vector has  $D-1$  off-shell and  $D-2$  on-shell dof. For the massless  $\Psi_\mu$  field we find here  $(D-1)2^{[\frac{D}{2}]}$  off-shell and  $\frac{1}{2}(D-3)2^{[\frac{D}{2}]}$  on-shell dof. The graviton field has  $\frac{1}{2}D(D-1)$  off-shell and  $\frac{1}{2}D(D-3)$  on-shell dof.

## 6 $\mathcal{N} = 1$ global supersymmetry in $D = 4$

The (classical) supersymmetry algebra is the completion of the Poincaré group with spinorial generators  $Q_\alpha$  that satisfy anticommutation relations

$$\{Q_\alpha, Q_\beta\} = -\frac{1}{2}(\gamma^\mu)_{\alpha\beta} P_\mu. \quad (6.1)$$

The supersymmetry transformations with constant spinor parameters can be written as  $\delta(\epsilon) = \bar{\epsilon}Q = \epsilon^\alpha Q_\alpha$ , and  $P_\mu$  is classically realized on fields as  $\partial_\mu$ . A basic realization of this algebra is the chiral multiplet with complex scalar fields  $Z$  and  $F$ , and Majorana spinor  $\chi$  (or its chiral projection  $P_L\chi$ ):

$$\delta(\epsilon)Z = \frac{1}{\sqrt{2}}\bar{\epsilon}P_L\chi, \quad \delta(\epsilon)P_L\chi = \frac{1}{\sqrt{2}}P_L(\not{\partial}Z + F)\epsilon, \quad \delta(\epsilon)F = \frac{1}{\sqrt{2}}\bar{\epsilon}\not{\partial}P_L\chi. \quad (6.2)$$

The simplest actions are those with kinetic terms and potential terms:

$$\begin{aligned} S &= S_{\text{kin}} + S_F + S_{\bar{F}}, \\ S_{\text{kin}} &= \int d^4x [-\partial^\mu \bar{Z} \partial_\mu Z - \bar{\chi} \not{\partial} P_L \chi + \bar{F} F], \quad S_F = \int d^4x [FW'(Z) - \frac{1}{2}\bar{\chi} P_L W''(Z)\chi], \end{aligned} \quad (6.3)$$

where  $F(Z)$  is a holomorphic function, called the superpotential.

We present the ‘gauge multiplet’ for non-Abelian gauge fields. The action, supersymmetry and gauge transformations are

$$\begin{aligned} S_{\text{gauge}} &= \int d^4x \left[ -\frac{1}{4}F^{\mu\nu A} F_{\mu\nu}^A - \frac{1}{2}\bar{\lambda}^A \gamma^\mu D_\mu \lambda^A + \frac{1}{2}D^A D^A \right], \\ \delta A_\mu^A &= -\frac{1}{2}\bar{\epsilon} \gamma_\mu \lambda^A + \partial_\mu \theta^A + \theta^C A_\mu^B f_{BC}^A, \\ \delta \lambda^A &= \left[ \frac{1}{4}\gamma^{\mu\nu} F_{\mu\nu}^A + \frac{1}{2}i\gamma_* D^A \right] \epsilon + \theta^C \lambda^B f_{BC}^A, \\ \delta D^A &= \frac{1}{2}i\bar{\epsilon} \gamma_* \gamma^\mu D_\mu \lambda^A + \theta^C D^B f_{BC}^A, \quad D_\mu \lambda^A \equiv \partial_\mu \lambda^A + \lambda^C A_\mu^B f_{BC}^A. \end{aligned} \quad (6.4)$$

$f_{BC}^A$  are the structure constants of the gauge group, which commutes with supersymmetry, but enters in the commutator of two supersymmetries:

$$[\delta(\epsilon_1), \delta(\epsilon_2)] = -\frac{1}{2}\bar{\epsilon}_1 \gamma^\nu \epsilon_2 \partial_\nu + \delta(\theta^A = \frac{1}{2}\bar{\epsilon}_1 \gamma^\nu \epsilon_2 A_\nu^A). \quad (6.5)$$

The right-hand side is called a ‘gauge-covariant translation’. The gauge multiplet can be coupled to chiral multiplets transforming in a representation of the gauge group. In that case, some supersymmetry transformations of the chiral multiplet are modified:

$$\delta(\epsilon)P_L\chi = \frac{1}{\sqrt{2}}P_L(\not{\partial}Z + F)\epsilon, \quad \delta(\epsilon)F = \frac{1}{\sqrt{2}}\bar{\epsilon}\not{\partial}P_L\chi - \bar{\epsilon}P_R\lambda^A t_A Z, \quad (6.6)$$

where  $\theta^A t_A Z$  is the gauge transformation of the scalar field, and  $D_\mu = \partial_\mu - A_\mu^A t_A$  are the gauge-covariant derivatives. Also in the action (6.3) ordinary derivatives are replaced by covariant derivatives, and supersymmetry requires an extra coupling term

$$S_{\text{coupling}} = \int d^4x \left[ -\sqrt{2}(\bar{\lambda}^A \bar{Z} t_A P_L \chi - \bar{\chi} P_R t_A Z \lambda^A) + i D^A \bar{Z} t_A Z \right]. \quad (6.7)$$

## 7 Differential geometry

For gravity we have to provide spacetime with a possibly nontrivial metric  $g_{\mu\nu}(x)$ . In any point of spacetime, this can be brought to a standard form using a ‘frame field’  $e_\mu^a(x)$ :

$$g_{\mu\nu}(x) = e_\mu^a(x) \eta_{ab} e_\nu^b(x). \quad (7.1)$$

In gravity theories with fermions, we have to make use of these frame fields. The Levi-Civita alternating symbol written in one or the other indices are related by

$$\begin{aligned} \varepsilon_{\mu_1 \mu_2 \dots \mu_D} &\equiv e^{-1} \varepsilon_{a_1 a_2 \dots a_D} e_{\mu_1}^{a_1} e_{\mu_2}^{a_2} \dots e_{\mu_D}^{a_D}, \\ \varepsilon^{\mu_1 \mu_2 \dots \mu_D} &\equiv e \varepsilon^{a_1 a_2 \dots a_D} e_{a_1}^{\mu_1} e_{a_2}^{\mu_2} \dots e_{a_D}^{\mu_D}, \end{aligned} \quad (7.2)$$

where  $e = \det e_\mu^a$ , while for all other vectors or tensors the components are related by equations of the type  $V_\mu = e_\mu^a V_a$ .

Differential forms offer often a convenient way to write field theories. We define the connection between components,  $p$ -forms and their exterior derivatives by

$$\omega^{(p)} = \frac{1}{p!} \omega_{\mu_1 \mu_2 \dots \mu_p} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \dots \wedge dx^{\mu_p}, \quad d\omega^{(p)} = \frac{1}{p!} \partial_\mu \omega_{\mu_1 \mu_2 \dots \mu_p} dx^\mu \wedge dx^{\mu_1} \wedge dx^{\mu_2} \wedge \dots \wedge dx^{\mu_p}. \quad (7.3)$$

The frame fields allow to write a local basis of 1-forms  $e^a \equiv e_\mu^a(x) dx^\mu$ . These are used to define the Hodge duals of  $p$ -forms as  $(D - p)$ -forms

$$*e^{a_1} \wedge \dots \wedge e^{a_p} = \frac{1}{p!} e^{b_1} \wedge \dots \wedge e^{b_q} \varepsilon_{b_1 \dots b_q}^{a_1 \dots a_p}. \quad (7.4)$$

These definitions lead to the expression

$$\int * \omega^{(p)} \wedge \omega^{(p)} = \frac{1}{p!} \int d^D x \sqrt{-g} \omega^{\mu_1 \dots \mu_p} \omega_{\mu_1 \dots \mu_p}, \quad (7.5)$$

which can be used to write a generalization of the Maxwell action as

$$S_p = -\frac{1}{2} \int * F^{(p+1)} \wedge F^{(p+1)}, \quad F^{(p+1)} \equiv dA^{(p)}. \quad (7.6)$$

The  $p = 1$  case is the Maxwell field. The field equations and Bianchi identities for  $F^{p+1}$  can be interpreted as Bianchi identities and field equations for  $*F^{p+1}$ , which can therefore be interpreted as the field strength of a  $(D - p - 2)$ -form. This implies that such actions for  $p$ -forms and for  $(D - p - 2)$ -forms are equivalent. This is the generalization of the electric-magnetic duality of Maxwell fields in 4 dimensions, where the electric 1-form is transformed to a magnetic 1-form. Further, for 4 dimensions it implies that an action (7.6) for an antisymmetric tensor (2-form) is equivalent to a scalar action. In higher dimensions, this is, however, not the case, and 3-forms are essential to construct  $D = 11$  supergravity.

For supergravity one must use covariant derivatives. They involve connections that are gauge fields for the Lorentz transformations, i.e.  $\omega_\mu^{ab} = -\omega_\mu^{ba}$  such that under the infinitesimal transformations of (1.2),  $\delta \omega_\mu^{ab} = \partial_\mu \lambda^{ab} - \lambda^a_c \omega_\mu^{cb} + \omega_\mu^{ac} \lambda_c^b$ , as appropriate for a gauge field for the algebra (1.3). Furthermore, there is an affine connection  $\Gamma_{\mu\nu}^\rho$  determined by the *vielbein postulate*

$$\nabla_\mu e_\nu^a = \partial_\mu e_\nu^a + \omega_\mu^a{}_b e_\nu^b - \Gamma_{\mu\nu}^\sigma e_\sigma^a = 0. \quad (7.7)$$

‘Torsion’ is the antisymmetric part of  $\Gamma$ . In form language

$$de^a + \omega^a{}_b \wedge e^b \equiv T^a = \frac{1}{2} T_{\mu\nu}^a dx^\mu \wedge dx^\nu, \quad \Gamma_{\mu\nu}^\rho - \Gamma_{\nu\mu}^\rho = T_{\mu\nu}^\rho. \quad (7.8)$$

We can split these connections in the metric-part and the torsion-dependent part:

$$\begin{aligned} \omega_\mu^{ab} &= \omega_\mu^{ab}(e) + K_\mu^{ab}, & \omega_\mu^{ab}(e) &= 2e^{\nu[a} \partial_{[\mu} e_{\nu]}^{b]} - e^{\nu[a} e^{b]\sigma} e_{\mu\sigma} \partial_\nu e_\sigma^c, \\ K_{\mu[\nu\rho]} &= -\frac{1}{2} (T_{[\mu\nu]\rho} - T_{[\nu\rho]\mu} + T_{[\rho\mu]\nu}), \\ \Gamma_{\mu\nu}^\rho &= \Gamma_{\mu\nu}^\rho(g) - K_{\mu\nu}^\rho, & \Gamma_{\mu\nu}^\rho(g) &= \frac{1}{2} g^{\rho\sigma} (\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}). \end{aligned} \quad (7.9)$$

The curvature tensor can be defined from both connections, and an integrability condition on (7.7) implies that these are related by changing indices in the conventional way:

$$\begin{aligned} R_{\mu\nu ab} &\equiv \partial_\mu \omega_{\nu ab} - \partial_\nu \omega_{\mu ab} + \omega_{\mu ac} \omega_{\nu}{}^c{}_b - \omega_{\nu ac} \omega_{\mu}{}^c{}_b, \\ R_{\mu\nu}{}^\rho{}_\sigma &\equiv \partial_\mu \Gamma_{\nu\sigma}^\rho - \partial_\nu \Gamma_{\mu\sigma}^\rho + \Gamma_{\mu\tau}^\rho \Gamma_{\nu\sigma}^\tau - \Gamma_{\nu\tau}^\rho \Gamma_{\mu\sigma}^\tau, & R_{\mu\nu}{}^\rho{}_\sigma &= R_{\mu\nu ab} e^{a\rho} e_\sigma^b. \end{aligned} \quad (7.10)$$

## 8 The first and second order formulations of general relativity

General relativity is well-known for bosonic fields, but there are several subtleties when fermions are involved. Let us consider

$$S = S_2 + S_{1/2} = \int d^D x e \left[ \frac{1}{2\kappa^2} e_a^\mu e_b^\nu R_{\mu\nu}{}^{ab} - \frac{1}{2} \bar{\Psi} \gamma^\mu \nabla_\mu \Psi + \frac{1}{2} \bar{\Psi} \overleftarrow{\nabla}_\mu \gamma^\mu \Psi \right], \quad (8.1)$$

where  $\kappa$  is the gravitational coupling constant,  $\gamma^\mu = e_a^\mu \gamma^a$  and

$$\nabla_\mu \Psi = (\partial_\mu + \frac{1}{4} \omega_\mu{}^{ab} \gamma_{ab}) \Psi, \quad \bar{\Psi} \overleftarrow{\nabla}_\mu = \bar{\Psi} (\overleftarrow{\partial}_\mu - \frac{1}{4} \omega_\mu{}^{ab} \gamma_{ab}), \quad (8.2)$$

are the Lorentz covariant derivatives. We consider here first that  $\omega_\mu{}^{ab}$  is only  $\omega_\mu{}^{ab}(e)$ , i.e. without torsion, and  $R_{\mu\nu}{}^{ab}$  is the expression (7.10) using  $\omega_\mu{}^{ab}(e)$ . The derivatives in (7.10) and (7.9) imply that the action is a second order action in the independent field  $e_\mu^a$ . Therefore, this is called ‘*second order formalism*’.

Another approach considers  $e_a^\mu$  and  $\omega_\mu{}^{ab}$  as independent fields. The action is then only first order in independent fields, and therefore this is called ‘*first order formalism*’. We consider first the field equation for  $\omega_\mu{}^{ab}$ . Without the fermionic part this would give  $\omega_\mu{}^{ab} = \omega_\mu{}^{ab}(e)$ . Therefore, comparing with (7.9) shows that the fermionic terms determine torsion. In fact, we get the solution as in (7.9) with

$$\omega_\mu{}^{ab} = \omega_\mu{}^{ab}(e) + K_\mu{}^{ab}, \quad K^\nu{}_{ab} = -\frac{1}{4} \kappa^2 \bar{\Psi} \gamma_{ab}{}^\nu \Psi, \quad T_{ab}{}^\nu = -2K^\nu{}_{ab}. \quad (8.3)$$

Thus, using this result, we have the same action as in (8.1), but with another definition of  $\omega_\mu{}^{ab}$ . To make the difference explicit, one may rewrite that action by expanding (8.3) and this gives

$$S = S(T=0) + \frac{1}{32} \kappa^2 \int d^D x e (\bar{\Psi} \gamma_{\mu\nu\rho} \Psi) (\bar{\Psi} \gamma^{\mu\nu\rho} \Psi), \quad (8.4)$$

where the first term is (8.1) with the torsionless connection. Due to the smallness of the gravitational coupling constant, the 4-fermion contact term cannot be measured in practice, but in principle there is a physical difference between this first and second order action. To obtain local supersymmetric actions, it turns out that in supergravity similar terms appear which can be understood from the first-order formalism [6].

## 9 Outlook

Supergravity is based on the ingredients mentioned in these lectures. In fact, the reader is now well equipped to study the first supergravity theories.

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